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On the Kinematics of Finite Strain Plasticity

by

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Abstract

In this paper, the representation of the kinematics of inelastic flow problems involving finite strains and rotations is discussed. The concept of multiplicatively decomposing the deformation gradient into elastic and plastic components is utilized, where the plastic deformation gradient represents a stress-free, relaxed configuration. Here, it is demonstrated that the choice of relaxed configuration is not essential in the problem solution. Therefore, the kinematic decomposition of elastic-plastic deformation may be chosen to best analyze the specific material model of concern. This is demonstrated by analyzing two specific materials, the planar single crystal and the glassy polymer, using various kinematic representations.

1 Introduction

The representation of inelastic material behavior under finite strains and rotations has been the subject of much discussion in the recent plasticity literature [FARDSHISHEH AND ONAT (1972), ONAT (1982), LORET (1983), DAFALLAS (1984, 1985), ANAND (1985), AGAH-TEHRANI, ET AL. (1987)]. Finite strain formulations generally begin with a description of the deformed body. This may be given by the deformation gradient, $\mathbf{F} = \nabla_{\mathbf{X}}\mathbf{x}$ where \mathbf{X} represents the reference position of a material point, \mathbf{x} represents the current position, and ∇ is the gradient. The multiplicative decomposition of the deformation gradient into elastic and plastic components was first proposed by LEE (1969), thereby introducing the concept of a relaxed, intermediate configuration represented by the plastic deformation gradient. Recently, the physical significance of such an intermediate configuration has been debated. HILL AND RICE (1972), ASARO (1983), HAVNER (1982) have proposed a convenient framework for the representation of single crystal plasticity where the rotation of the crystal lattice is perceived to be an elastic deformation. The plastic deformation gradient represents the relaxed configuration such that the crystal is stress-free and the lattice is in its reference orientation.

2 Kinematics of the Single Crystal

The basic kinematics of the representation of inelastic behavior under finite strains and rotations for a single crystal are presented. In this section, two "elastic"- "plastic" multiplicative decompositions of the deformation gradient will be discussed. The first representation (referred to as *R-I*) obtains the relaxed configuration by elastically unloading to a stress free state while the crystal lattice reorients itself with its initial reference configuration. In *R-I*, the relaxed configuration is, in effect, prescribed by constitutive constraints on the plastic spin. The second representation (*R-II*) obtains a relaxed configuration by elastically unloading to a stress free state without any rotation. This requires the "elastic" deformation gradient to be symmetric, i.e. a stretch tensor. The spin of the relaxed configuration of *R-II* will be shown to be algebraically prescribed due to the symmetry restriction on its "elastic" deformation gradient. These representations are discussed in detail below for the specific case of the single crystal. Terms which are specific to *R-I* will be subscripted *I*; terms which are specific to the *R-II* will be subscripted *II*. Terms which are common to both, such as the total deformation gradient, \mathbf{F} , and the velocity gradient, \mathbf{L} , will not be subscripted.

Some basic kinematic definitions and terminology are now given. As discussed in the introduction, a material point in its original undeformed configuration will be represented by \mathbf{X} . The same material point in the deformed configuration will be represented by \mathbf{x} . The deformed body may be described by its deformation gradient, \mathbf{F} , which is given by:

$$\mathbf{F} = \nabla_{\mathbf{X}} \mathbf{x}. \quad (1)$$

The corresponding rate kinematics begin with the velocity gradient, \mathbf{L} , which is given by:

$$\mathbf{L} = \nabla_{\mathbf{x}} \mathbf{v} = \dot{\mathbf{F}} \mathbf{F}^{-1} = \mathbf{D} + \mathbf{W}; \quad (2)$$

where \mathbf{D} , the rate of deformation, is the symmetric part of \mathbf{L} ; and \mathbf{W} , the spin, is the skew part of \mathbf{L} .

2.1 Basic Kinematics of Representation I

The kinematics which are specific to Representation I are now given, beginning with the multiplicative decomposition of the deformation gradient [ASARO (1983)]:

$$\mathbf{F} = \mathbf{F}_I^* \mathbf{F}_I^p; \quad (3)$$

$$\mathbf{F}_I^* = \mathbf{F}_I^* \mathbf{F}_I^R; \quad (4)$$

where \mathbf{F}_I^p is the deformation gradient due to plastic slip; \mathbf{F}_I^R is the residual rotation of the lattice; and \mathbf{F}_I^* is the elastic deformation gradient. In other terms, the gradient \mathbf{F}_I^* accounts for the stretching and rotation of the lattice, while \mathbf{F}_I^p accounts for plastic shearing on the crystal slip planes of fixed (reference) orientation.

The rate kinematics may also be decomposed into "elastic" and "plastic" components. This is done by substituting the decomposition given in equation (3) into equation (2), from which we obtain:

$$\begin{aligned} \mathbf{L} &= \dot{\mathbf{F}}_I^* \mathbf{F}_I^{*-1} + \mathbf{F}_I^* \dot{\mathbf{F}}_I^p \mathbf{F}_I^{p-1} \mathbf{F}_I^{*-1}; \\ \mathbf{L}_I^p &= \mathbf{F}_I^* \dot{\mathbf{F}}_I^p \mathbf{F}_I^{p-1} \mathbf{F}_I^{*-1} = \mathbf{D}_I^p + \mathbf{W}_I^p; \end{aligned} \quad (5)$$

where \mathbf{L}_I^p is the velocity gradient of the relaxed configuration convected to the deformed configuration, \mathbf{F} ; \mathbf{D}_I^p and \mathbf{W}_I^p are the rate of plastic deformation and the plastic spin, respectively. These last two terms, \mathbf{D}_I^p and \mathbf{W}_I^p , must both be constitutively prescribed in this formulation, which is precisely what is done immediately below for the case of the single crystal.

Constitutive Law for \mathbf{D}_I^p and \mathbf{W}_I^p

A crystal slip system may be defined by the dyadic $\mathbf{s}^\alpha \otimes \mathbf{m}^\alpha$, where the vector \mathbf{s}^α lies along the α slip plane and the vector \mathbf{m}^α is perpendicular to this slip plane. The velocity gradient in the relaxed configuration of representation I is given by $\tilde{\mathbf{L}}_I^p = \dot{\mathbf{F}}_I^p \mathbf{F}_I^{p-1}$. This is constitutively obtained by the summation of the rate of plastic straining, $\dot{\gamma}^\alpha$, of each slip system ($\mathbf{s}^\alpha \otimes \mathbf{m}^\alpha$):

$$\tilde{\mathbf{L}}_I^p = \sum_{\alpha} \dot{\gamma}^\alpha \mathbf{s}^\alpha \otimes \mathbf{m}^\alpha. \quad (6)$$

The plastic velocity gradient in the loaded configuration is obtained by convecting the slip system to this configuration:

$$\mathbf{s}^{*\alpha} \otimes \mathbf{m}^{*\alpha} = \mathbf{F}_I^* (\mathbf{s}^\alpha \otimes \mathbf{m}^\alpha) \mathbf{F}_I^{*-1}; \quad (7)$$

resulting in the following expression for \mathbf{L}_I^p of the loaded state:

$$\mathbf{L}_I^p = \sum_{\alpha} \dot{\gamma}^\alpha \mathbf{s}^{*\alpha} \otimes \mathbf{m}^{*\alpha}. \quad (8)$$

The rate of plastic deformation and the plastic spin in the loaded and unloaded configurations may be obtained by first decomposing the plastic slip direction tensors $s^a \otimes m^a$ and $s^{*a} \otimes m^{*a}$ into their symmetric and antisymmetric parts, which we define by:

$$\begin{aligned}\tilde{S}^a &= \frac{1}{2}(s^a \otimes m^a + m^a \otimes s^a); \\ \tilde{A}^a &= \frac{1}{2}(s^a \otimes m^a - m^a \otimes s^a); \\ S^a &= \frac{1}{2}(s^{*a} \otimes m^{*a} + m^{*a} \otimes s^{*a}); \\ A^a &= \frac{1}{2}(s^{*a} \otimes m^{*a} - m^{*a} \otimes s^{*a});\end{aligned}\tag{9}$$

giving:

$$\begin{aligned}D_I^p &= \sum_a \dot{\gamma}^a S^a; \\ W_I^p &= \sum_a \dot{\gamma}^a A^a; \\ \tilde{D}_I^p &= \sum_a \dot{\gamma}^a \tilde{S}^a; \\ \tilde{W}_I^p &= \sum_a \dot{\gamma}^a \tilde{A}^a.\end{aligned}\tag{10}$$

The rates D_I^p and W_I^p have now been constitutively prescribed in terms of the plastic strain rates of the individual slip systems. Later in this paper, a specific problem considering a "planar" single crystal with two slip systems and a viscoplastic constitutive law for $\dot{\gamma}^a$ will be presented. For the present, we can see that this first representation accounts for the material and lattice deformation of the crystal.

2.2 Basic Kinematics of Representation II

The kinematics specific to Representation II will now be given. The multiplicative decomposition of R_{II} is given by:

$$F = F_{II}^e F_{II}^p;\tag{11}$$

$$F_{II}^{eT} = F_{II}^e;\tag{12}$$

where F_{II}^e is the symmetric elastic deformation gradient (an elastic stretch tensor); and F_{II}^p is the deformation gradient of the relaxed configuration obtained by elastically unloading without rotation to a stress free state. In comparison with F_I^p which describes the plastic shearing of the material, F_{II}^p describes the plastic shearing as well as any (residual or elastic) lattice rotations.

The velocity gradient is decomposed via incorporation of equation (11) into equation (2), giving:

$$\begin{aligned} \mathbf{L} &= \dot{\mathbf{F}}_{II}^e \mathbf{F}_{II}^{e-1} + \mathbf{F}_{II}^e \dot{\mathbf{F}}_{II}^p \mathbf{F}_{II}^{p-1} \mathbf{F}_{II}^{e-1}; \\ \mathbf{L}_{II}^p &= \dot{\mathbf{F}}_{II}^p \mathbf{F}_{II}^{p-1} = \mathbf{D}_{II}^p + \mathbf{W}_{II}^p; \end{aligned} \quad (13)$$

where \mathbf{L}_{II}^p is the velocity gradient of the relaxed configuration of $R-II$, \mathbf{F}_{II}^p ; \mathbf{D}_{II}^p is the rate of deformation of the relaxed configuration which must be constitutively prescribed; and \mathbf{W}_{II}^p is the spin of the relaxed configuration. The spin, \mathbf{W}_{II}^p , is algebraically prescribed due to the symmetry restriction on \mathbf{F}_{II}^e . This is shown immediately below:

$$\begin{aligned} \dot{\mathbf{F}}_{II}^e &= (\mathbf{D} + \mathbf{W})\mathbf{F}_{II}^e - \mathbf{F}_{II}^e(\mathbf{D}_{II}^p + \mathbf{W}_{II}^p), \\ \text{and,} \\ \dot{\mathbf{F}}_{II}^e &= (\dot{\mathbf{F}}_{II}^e)^T = \mathbf{F}_{II}^e(\mathbf{D} - \mathbf{W}) - (\mathbf{D}_{II}^p - \mathbf{W}_{II}^p)\mathbf{F}_{II}^e; \\ \text{or,} \\ (\mathbf{D} + \mathbf{D}_{II}^p)\mathbf{F}_{II}^e - \mathbf{F}_{II}^e(\mathbf{D} + \mathbf{D}_{II}^p) &= \mathbf{F}_{II}^e(\mathbf{W}_{II}^p - \mathbf{W}) + (\mathbf{W}_{II}^p - \mathbf{W})\mathbf{F}_{II}^e; \end{aligned} \quad (14)$$

therefore, \mathbf{W}_{II}^p is algebraically obtained to be:

$$\mathbf{W}_{II}^p = \mathbf{W} - \mathcal{W}_{II}[\mathbf{D} + \mathbf{D}_{II}^p]; \quad (15)$$

where \mathcal{W}_{II} is a fourth order tensor with components of order elastic strain ONAT (1987) mapping symmetric second order tensors to skew tensors. Therefore, \mathbf{W}_{II}^p must not be constitutively prescribed.

Thus far, in the context of the second representation, no explicit provision has been made for monitoring the orientation of the crystal lattice. Onat has shown theoretically that $R-II$ can accommodate the internal state and orientation of a material element by the updating of an appropriate set of irreducible even rank tensor state variables ONAT (1982). Here, we will carry out this proposal for the problem of the single crystal.

In $R-II$, the deformation gradient of the relaxed configuration contains both the material and lattice rotations at a material point. Therefore, for the case of the single crystal, \mathbf{F}_{II}^p may be decomposed into the product of lattice rotation \mathbf{R}^L and plastic slip, \mathbf{F}^{slip} :

$$\mathbf{F}_{II}^p = \mathbf{R}^L \mathbf{F}^{slip}. \quad (16)$$

Here, \mathbf{F}^{slip} describes the deformation due to shearing on slip planes in the reference orientation and is equivalent to \mathbf{F}_I^p of $R-I$. The crystal slip system in the relaxed configuration is given by:

$$\mathbf{s}_{II}^a \otimes \mathbf{m}_{II}^a = \mathbf{R}^L \mathbf{s}^a \otimes \mathbf{R}^L \mathbf{m}^a; \quad (17)$$

where \mathbf{s}^a and \mathbf{m}^a describe the initial directions of each slip plane and its normal. The slip system may be further decomposed into symmetric and skew parts:

$$\mathbf{S}_{II}^a = \frac{1}{2}[\mathbf{s}_{II}^a \otimes \mathbf{m}_{II}^a + \mathbf{m}_{II}^a \otimes \mathbf{s}_{II}^a]; \quad (18)$$

$$\mathbf{A}_{II}^a = \frac{1}{2}[\mathbf{s}_{II}^a \otimes \mathbf{m}_{II}^a - \mathbf{m}_{II}^a \otimes \mathbf{s}_{II}^a]. \quad (19)$$

The tensors \mathbf{S}_{II}^a and \mathbf{A}_{II}^a are the state variables needed in $R-II$ to appropriately monitor the lattice orientation. It is emphasized that \mathbf{S}_{II}^a and \mathbf{A}_{II}^a provide an irreducible set of even ranked tensors which monitor the lattice, where \mathbf{S}_{II}^a is a symmetric traceless 2nd order tensor and \mathbf{A}_{II}^a is a 2nd order skew tensor. The rate of change of these state variables is found to be given by:

$$\dot{\mathbf{S}}_{II}^a = [\dot{\mathbf{R}}^L(\mathbf{R}^L)^T] \mathbf{S}_{II}^a + \mathbf{S}_{II}^a [\dot{\mathbf{R}}^L(\mathbf{R}^L)^T]^T; \quad (20)$$

$$\dot{\mathbf{A}}_{II}^a = [\dot{\mathbf{R}}^L(\mathbf{R}^L)^T] \mathbf{A}_{II}^a + \mathbf{A}_{II}^a [\dot{\mathbf{R}}^L(\mathbf{R}^L)^T]^T; \quad (21)$$

where $\dot{\mathbf{R}}^L(\mathbf{R}^L)^T$ is the lattice spin.

The velocity gradient of the relaxed configuration Π , \mathbf{L}_{II}^p , may be expanded, using equation (16), as:

$$\mathbf{L}_{II}^p \equiv \dot{\mathbf{F}}_{II}^p \mathbf{F}_{II}^{p-1} = \dot{\mathbf{R}}^L(\mathbf{R}^L)^T + \mathbf{R}^L \dot{\mathbf{F}}^{slip} (\mathbf{F}^{slip})^{-1} (\mathbf{R}^L)^T, \quad (22)$$

where we know the velocity gradient due to plastic slipping is constitutively prescribed by the rate of shearing on the slip systems. Therefore, using (22) and (18, 19), the rate of deformation and the spin of relaxed configuration Π , can be expressed as:

$$\mathbf{D}_{II}^p = \sum_a \dot{\gamma}^a \mathbf{S}_{II}^a; \quad (23)$$

and

$$\mathbf{W}_{II}^p = \dot{\mathbf{R}}^L(\mathbf{R}^L)^T + \sum_a \dot{\gamma}^a \mathbf{A}_{II}^a. \quad (24)$$

Since \mathbf{W}_{II}^p is algebraically defined via equation (15), we obtain an expression for the lattice spin from equation (24):

$$\Omega^L \equiv \dot{\mathbf{R}}^L(\mathbf{R}^L)^T = \mathbf{W}_{II}^p - \sum_a \dot{\gamma}^a \mathbf{A}_{II}^a. \quad (25)$$

Equations (20), (21), and (25) may then be used to update the state. The state variables S_{II} and A_{II} may be updated via an incremental orthogonal transformation, Q , corresponding to $\Omega^L \Delta t$, where the corotational rates of S_{II} and A_{II} are given by:

$$\dot{S}_{II}^a = \dot{S}_{II}^a - \Omega^L S_{II}^a + S_{II}^a \Omega^L = 0, \quad (26)$$

$$\dot{A}_{II}^a = \dot{A}_{II}^a - \Omega^L A_{II}^a + A_{II}^a \Omega^L = 0, \quad (27)$$

and the "intermediate" rotation, Q , must therefore satisfy:

$$\dot{Q} = \Omega_L Q, \quad (28)$$

$$Q(t) = I. \quad (29)$$

The state variables S_{II}^a and A_{II}^a are then updated via:

$$(S_{II}^a)_{t+\Delta t} = Q_{t+\Delta t} (S_{II}^a)_t Q_{t+\Delta t}^T, \quad (30)$$

$$(A_{II}^a)_{t+\Delta t} = Q_{t+\Delta t} (A_{II}^a)_t Q_{t+\Delta t}^T. \quad (31)$$

The preceding equations are easier to visualize if we simplify to the case of no elastic stretching. In this case, the expressions in terms of the first representation, where F_I^* is now simply a rotation R^* , become:

$$\begin{aligned} F &= R^* F_I^p; \\ L &= \dot{R}^* R^{*T} + R^* \dot{F}_I^p F_I^{p-1} R^{*T}; \\ D &= R^* \dot{D}_I^p R^{*T}; \\ W &= \dot{R}^* R^{*T} + R^* \dot{W}_I^p R^{*T}. \end{aligned} \quad (32)$$

The expressions in terms of the second representation, where $F_{II}^e = I$, become:

$$\begin{aligned} F &= F_{II}^p; \\ L &= L_{II}^p; \\ D &= D_{II}^p = \sum_a \dot{\gamma}^a S_{II}^a; \\ W &= W_{II}^p; \\ \Omega^L &= W - \sum_a \dot{\gamma}^a A_{II}^a. \end{aligned} \quad (33)$$

For completeness, the kinematic relationship between representations I and II is now given. We first pictorially examine a body as it goes from its original undeformed configuration to a deformed configuration as shown in Figure 1. Here, the "relaxed" states as defined by F_{II}^p and F_I^p are also depicted. Relaxed configuration I is a stress free

state containing no lattice deformation or changes in orientation. Relaxed configuration II is the stress free state which contains any changes in lattice orientation.

Once again, we examine the elastic-plastic decomposition of the deformation gradient as implemented in *R-I*, $\mathbf{F} = \mathbf{F}_I^* \mathbf{F}_I^p$. The lattice deformation gradient, \mathbf{F}_I^* , may be broken down into its left stretch and rotation tensors via the polar decomposition theorem to give:

$$\mathbf{F}_I^* = \mathbf{V}^* \mathbf{R}^*; \quad (34)$$

$$\mathbf{F} = \mathbf{V}^* \mathbf{R}^* \mathbf{F}_I^p; \quad (35)$$

where \mathbf{V}^* describes the elastic stretch of the material. The decomposition $\mathbf{F} = \mathbf{F}_{II}^e \mathbf{F}_{II}^p$, where \mathbf{F}_{II}^e is a symmetric stretch tensor, yields a unique \mathbf{F}_{II}^e . Therefore, since both \mathbf{V}^* and \mathbf{F}_{II}^e describe the elastic stretch incurred by the solid, we must conclude that:

$$\mathbf{V}^* = \mathbf{F}_{II}^e. \quad (36)$$

In other words, if we elastically unload to a stress-free configuration via $\mathbf{F}^{*-1} = \mathbf{F}_{II}^{e-1}$, then \mathbf{F}_{II}^e is unique. Therefore, if we elastically unload to a stress-free state via \mathbf{V}^{*-1} , it must be the same stress-free state described by \mathbf{F}_{II}^e . The rotation matrix \mathbf{R}^{*T} retains an unloaded stress-free configuration and simply rotates the lattice to its appropriate stress-free configuration which would be its initial undeformed configuration. Therefore, the following relation is obtained:

$$\mathbf{F}_{II}^p = \mathbf{R}^* \mathbf{F}_I^p; \quad (37)$$

or, alternatively,

$$\mathbf{F}_I^p = \mathbf{R}^{*T} \mathbf{F}_{II}^p; \quad (38)$$

where \mathbf{R}^* is the rotation tensor which defines the lattice orientation. Therefore, it is clear that \mathbf{R}^* and \mathbf{R}^L are identical tensors. The relationship between the rate quantities of *R-I* and *R-II* is also established:

$$\mathbf{L}_{II}^p = \mathbf{\Omega}^L + \mathbf{R}^L \tilde{\mathbf{L}}_I^p \mathbf{R}^{LT}; \quad (39)$$

$$\mathbf{W}_{II}^p = \mathbf{\Omega}^L + \mathbf{R}^L \tilde{\mathbf{W}}_I^p \mathbf{R}^{LT}; \quad (40)$$

$$\mathbf{D}_{II}^p = \mathbf{R}^L \tilde{\mathbf{D}}_I^p \mathbf{R}^{LT}; \quad (41)$$

A physical interpretation of the above equation is that the $\mathbf{R}^L \tilde{\mathbf{L}}^p \mathbf{R}^{LT}$ rotates the velocity gradient of the *R-I* unloaded configuration into the *R-II* unloaded configuration. The Ω^L accounts for the rate of this rotation and is the spin of the lattice. In other words, \mathbf{R}^L is effectively a time-dependent observer transformation relating two different elastically unloaded configurations.

2.3 Example Problem: Planar Single Crystal

The problem of the planar single crystal with two shear slip systems and no elastic stretching is now examined. This problem has been analyzed by PIERCE, ASARO, AND NEEDLEMAN (1983). For completeness and for comparative purposes, their solution will be repeated here. A solution for this same problem using the second representation will also be presented.

We begin with some definitions of basic parameters to be used in the problem as illustrated in Figure 2. The primary and conjugate slip systems are defined by the vector pairs $(\mathbf{s}^p, \mathbf{m}^p)$ and $(\mathbf{s}^c, \mathbf{m}^c)$, respectively. The position of the primary system is given by the angle ϕ clockwise from the axis of imposed tensile deformation, \mathbf{e}_2 . The position of the conjugate system is given by the angle ψ counterclockwise from the axis of imposed tensile deformation, \mathbf{e}_2 . Initial conditions on these angles were taken to be: $\phi(0) = 40^\circ$ and $\psi(0) = 20^\circ$. The angle β in the figure describes the orientation of the lattice with initial condition of $\beta(0) = 0$. Therefore, the lattice orientation tensor \mathbf{R}^* is found to be:

$$\mathbf{R}^* = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}. \quad (42)$$

The rate of plastic shear straining on the primary and conjugate slip systems is given by $\dot{\gamma}^p$ and $\dot{\gamma}^c$, respectively. The visco-plastic constitutive law for these $\dot{\gamma}^a$ is given by the power-law relation:

$$\dot{\gamma}^a = \dot{a}^a \left(\frac{\tau^a}{g^a} \right)^m; \quad (43)$$

where \dot{a}^a is a parameter such that $\dot{\gamma}^a = \dot{a}^a$ when $\tau^a = g^a$; τ^a is the resolved shear stress on that slip system; g^a represents the internal structure of the material and may evolve with strain hardening, whereupon the g^a become internal state variables which monitor the hardness of each slip system; and m indicates the rate sensitivity of the material. The internal structure is taken to evolve according to:

$$\dot{\gamma}^a = \sum_{\beta} h_{a\beta} \dot{\gamma}^{\beta}; \quad (44)$$

where $h_{a\beta}$ is a hardness slope matrix and therefore accounts for latent as well as self strain hardening. Following PIERCE, ET AL., for this example, we take:

$$\begin{aligned} h_{pc} &= qh_{pp}; & h_{cp} &= qh_{cc}; \\ h_{pp} &= h; & h_{cc} &= h; \end{aligned} \quad (45)$$

where,

$$h = h_0 \operatorname{sech}^2 \left(\frac{h_0(\gamma^p + \gamma^c)}{\tau_s - \tau_0} \right); \quad (46)$$

where, for aluminum 2.8 wt.% copper alloys, $h_0 = 8.9\tau_0$ and $\tau_s = 1.8\tau_0$ and $1.0 \leq q \leq 1.4$ PIERCE, ET AL..

2.3.1 Solution Using Representation I

The relevant material law information has now been given and we now move onto the solution of pulling this single crystal in tension along the x_2 axis. Since the crystal is subjected to simple tension, points which lie along the x_2 axis will remain along this axis giving:

$$\mathbf{F}\mathbf{e}_2 = \lambda \mathbf{e}_2 \quad (47)$$

where λ is the stretch in this direction. Using representation I results in:

$$\tan \beta = F_{I12}^p / F_{I22}^p \quad (48)$$

and

$$\dot{\beta} = \cos^2 \beta \left[\frac{\dot{F}_{I12}^p}{F_{I22}^p} - \frac{F_{I12}^p \dot{F}_{I22}^p}{(F_{I22}^p)^2} \right]; \quad (49)$$

where \dot{F}_I^p is obtained from:

$$\dot{F}_I^p = [\dot{\gamma}^p \mathbf{s}^p \otimes \mathbf{m}^p + \dot{\gamma}^c \mathbf{s}^c \otimes \mathbf{m}^c] \mathbf{F}_I^p. \quad (50)$$

These equations, together with the resolved shear stresses

$$\begin{aligned} \tau^p &= \frac{\sigma}{2} \sin(2\phi - 2\beta); \\ \tau^c &= \frac{\sigma}{2} \sin(2\psi + 2\beta); \end{aligned} \quad (51)$$

are used to update the state of the material. The orientation of the lattice, β , as a function of the amount of plastic strain on the primary system was found for the following three cases (all with $\dot{\alpha} = 10^{-2} \text{sec}^{-1}$): $m = 50, q = 1.4$; $m = 50, q = 1.0$; $m = 10, q = 1.0$. The results are shown in Figure 3 and were originally calculated by PIERCE, ET AL.. We note here that β is obtained from \mathbf{F}_I^p by equation (48).

2.3.2 Solution Using Representation II

Kinematic representation II will now be used to analyze the "planar" single crystal, where the lattice orientation is included as a state variable by integrating equations (20) and (21) as suggested in section 2.2.

Since we are considering an elastically rigid single crystal, $\mathbf{F}_{II}^e = \mathbf{I}$, and the plastic deformation gradient is equal to the total deformation gradient:

$$\mathbf{F}_{II}^p = \mathbf{F}. \quad (52)$$

Similarly, the plastic spin is equal to the total spin:

$$\mathbf{W}_{II}^p = \mathbf{W} = \text{skew}[\dot{\mathbf{F}}\mathbf{F}^{-1}] = \text{skew}[\dot{\mathbf{F}}_{II}^p\mathbf{F}_{II}^{p-1}]. \quad (53)$$

This spin does not account for the lattice orientation. The lattice spin is given by:

$$\Omega_L = [\mathbf{W}_{II}^p - \dot{\gamma}^p \mathbf{A}_{II}^p - \dot{\gamma}^e \mathbf{A}_{II}^e]. \quad (54)$$

The lattice orientation may then be updated with equations (30) and (31) with the initial conditions:

$$(\mathbf{S}_{II}^a)_{t=0} = \frac{1}{2}[\mathbf{s}^a \otimes \mathbf{m}^a + \mathbf{m}^a \otimes \mathbf{s}^a], \quad (55)$$

$$(\mathbf{A}_{II}^a)_{t=0} = \frac{1}{2}[\mathbf{s}^a \otimes \mathbf{m}^a - \mathbf{m}^a \otimes \mathbf{s}^a], \quad (56)$$

Since this is a planar problem, the lattice orientation may also be found by simply updating the angle β , where $\dot{\beta} = \Omega_{L21}$. These equations together with the constitutive laws (40 - 43) stated earlier are used to update the state of the material. The solution for the lattice orientation, β , as a function of the plastic strain on the primary slip system, γ^p , was found for the same three cases as the earlier solution. The results are plotted along with the earlier results in Figure 3, and with which they are identical.

It is important to note that the solution processes using the two different representations are very similar. However, they are operationally different in the manner in which the lattice orientation is taken into account. The kinematics of the elastic-plastic decomposition of the deformation gradient in representation I specifically includes the lattice orientation. The second kinematical decomposition is more general and does not explicitly contain the lattice orientation, which must be implicitly retained and "constitutively" updated as a tensor state variable. Therefore, this representation requires an additional tensor to properly describe the material behavior of a single crystal. In *R-I*, a description of the lattice orientation is not given, i.e. knowledge of the plastic deformation gradient is needed. However, in *R-II*, the complete state of the material at any point in time is completely described by the instantaneous hardnesses of the slip systems and the lattice orientation. In the next section, the case of the glassy polymer is examined in the context of different kinematical decompositions.

3 Kinematics of the Glassy Polymer

Below, we will give a brief outline of the constitutive model for the large inelastic deformation behavior of glassy polymers proposed in PARKS, ET AL. and BOYCE, ET AL.. The reference configuration of the glassy polymer is an isotropic state of the material which consists of randomly oriented chains (Figure 4). We assume that, during plastic flow, these molecules deform affinely. The relaxed configuration, F^p , is obtained by elastically unloading to a stress free state, and physically indicates the degree of permanent molecular orientation existing in the material. In general, we can express the total deformation gradient as the product of the elastic stretch, a rotation, and the plastic stretch:

$$F = V^e R U^p. \quad (57)$$

The rotation tensor R may be decomposed into the product of elastic and plastic tensors, $R = R^e R^p$, where $F^e = V^e R^e = R^e U^e$, and $F^p = R^p U^p = V^p R^p$, via the polar decomposition theorem. However, the "elasticity" and/or "plasticity" of the rotation tensor is indeterminate. If $R = R^p$ and $R^e = I$ is chosen, all such rotation effects are lumped into the affine plastic deformation response of the material. This results in a symmetric and, therefore, unique elastic deformation gradient, $F^{eT} = F^e$. This formulation is kinematic representation II discussed in section 2 above. Alternatively,

if $\mathbf{R} = \mathbf{R}^e$ and $\mathbf{R}^p = \mathbf{I}$ is chosen, \mathbf{F}^e is again unique, but not symmetric. A third alternative for rendering this factorization unique is to obtain the elastic-plastic decomposition of the rotation tensor by *imposing a constraint on the spin of the relaxed state*. Indeed, this was precisely the procedure followed in representation I above for the single crystal, where the plastic spin was constitutively prescribed, thereby generating a particular relaxed configuration.

In the remainder of the discussion on the glassy polymer, the modelling of the material behavior will be considered using two of the above kinematical representations: 1) a relaxed state which contains all rotations is chosen, i.e. $\mathbf{R}^p = \mathbf{R}$; 2) a relaxed state is chosen via a particular restriction on the plastic spin. The first choice, *R-II*, was used in PARKS, ET AL. and BOYCE, ET AL. In this case, the plastic spin is algebraically defined as discussed in section 2. The alternative of defining a relaxed configuration by imposing a constraint on the plastic spin will be made clearer after further discussion of the material model using the kinematic representation where $\mathbf{R}^p = \mathbf{R}$.

3.1 Choice 1: $\mathbf{R}^p = \mathbf{R}$

For the case where we select $\mathbf{R}^p = \mathbf{R}$, where \mathbf{R} is defined in equation (57), and $\mathbf{F}^e = \mathbf{F}^{eT}$, it was shown earlier that the plastic spin is algebraically defined via equation (15). A rate of plastic shape change must be constitutively prescribed for the material. We note that \mathbf{D}_{II}^p describes the rate of shape change of the relaxed configuration given by the kinematics \mathbf{F}_{II}^p . The magnitude of \mathbf{D}_{II}^p is given by the plastic shear strain rate, $\dot{\gamma}^p$, and the tensor direction of \mathbf{D}_{II}^p is specified by $\mathbf{N} = \frac{1}{\sqrt{3}} \mathbf{T}^*$, the normalized driving stress state, \mathbf{T}^* , at a material point. The driving stress state is given by:

$$\mathbf{T}^* = \mathbf{T} - \frac{1}{J} \mathbf{F}_{II}^e \tilde{\mathbf{B}} \mathbf{F}_{II}^{eT}, \quad (58)$$

where: \mathbf{T} is the Cauchy stress tensor given by $\mathbf{T} = \frac{1}{J} \mathcal{L}^e [\ln \mathbf{V}^e]$ [ANAND (1979)], \mathcal{L}^e is the fourth order isotropic elasticity tensor, and \mathbf{V}^e is the left elastic stretch tensor; J is the volume change given by $\det \mathbf{F}_{II}^e$; and $\tilde{\mathbf{B}}$ is the back stress tensor resulting from strain hardening. The back stress is uniquely related to the inelastic distortion in the polymer, where $\tilde{\mathbf{B}}$ is coaxial with the left plastic stretch tensor:

$$\mathbf{F}_{II}^p = \mathbf{V}_{II}^p \mathbf{R}_{II}^p \quad \text{polar decomposition,} \quad (59)$$

$$\mathbf{V}_{II}^p = \tilde{\mathbf{Q}}_{II}^{pT} \bar{\mathbf{A}}^p \tilde{\mathbf{Q}}_{II}^p \quad \text{eigenvalue decomposition,} \quad (60)$$

$$B_i = B_i(\Lambda_i^p), \quad (61)$$

where B_i are the principal values of $\tilde{\mathbf{B}}$ defined in BOYCE, ET AL., and Λ_i^p are the principal values of Λ^p . The coaxiality of $\tilde{\mathbf{B}}$ and \mathbf{V}_{II}^p is a key result of the assumption that the molecules deform affinely during plastic flow. The effective equivalent shear stress, τ , is given by:

$$\tau = \left[\frac{1}{2} \mathbf{T}'' \cdot \mathbf{T}'' \right]^{\frac{1}{2}}. \quad (62)$$

We point out that this is essentially a kinematic-hardening law. The plastic shear strain rate is constitutively prescribed and may be functionally given by:

$$\dot{\gamma}^p = \dot{\gamma}^p(\tau, s, p, \Theta), \quad (63)$$

where s is the scalar athermal shear resistance of the material, which evolves with plastic straining; $p = -\frac{1}{3} \text{tr} \mathbf{T}$ is the pressure; and Θ is the absolute temperature. A detailed expression for $\dot{\gamma}^p$ is given in BOYCE, ET AL..

This has been a very brief description of the kinematic hardening model for the large inelastic deformation of glassy polymers proposed in PARKS, ET AL. and BOYCE, ET AL. The assumption of affine response of the molecular chains, which lead to the back stress coaxiality with the plastic stretch, was instrumental in selecting the kinematic representation of deformation. When the driving stress state is obtained, the back stress, which is computed in the relaxed configuration, is transformed to the loaded configuration. Therefore, alternative relaxed configurations which contain different degrees of rotations, i.e. $\mathbf{R}^p \neq \mathbf{R}$, should indeed result in the same material response, providing the alternative measures of back stress thus obtained are also consistently transformed to the loaded configuration. As stated earlier, another unique choice of kinematic decomposition is obtained by taking $\mathbf{R}^e = \mathbf{R}$ and $\mathbf{R}^p = \mathbf{I}$ and, therefore, $\mathbf{F}^p = \mathbf{U}^p$. The resulting principal values of the back stress tensor would be identical, but the tensor would be coaxial with \mathbf{U}^p and would have to be transformed via \mathbf{R}^e when computing the driving stress state \mathbf{T}^* . This decomposition would also sufficiently complicate the rate kinematics so as to make it an unreasonable choice for this material. A more reasonable alternate choice is to impose a constraint on \mathbf{W}^p , which is discussed next.

3.2 Choice 2: Prescribe W^P

We may also define the kinematics of the glassy polymer in a manner similar to representation I of the single crystal where the relaxed configuration, F_I^P , was chosen via a constraint on both the rate of plastic deformation D_I^P and the plastic spin W_I^P . In that case, the constraint was constitutive. For the case of the glassy polymer, where the molecules deform affinely, there is no obviously analogous or intuitive choice for a "constitutive" constraint on the spin of the relaxed configuration. Therefore, we may obtain our relaxed state represented by F^P by choosing a convenient W^P . Indeed, note the "convenience" offered in *R-I* by the imposition of the particular constraint on the spin of the relaxed configuration of the single crystal. The simplest possible form for the spin of the relaxed state is chosen:

$$W_{III}^P \equiv 0. \quad (64)$$

In other words, a relaxed configuration which is not spinning is chosen. In general, as in *R-I*, this will result in elastic and plastic deformation gradients which both contain rotations, $F = F_{III}^e F_{III}^P = V^e R_{III}^e R_{III}^P U^P$. The rate of deformation of the relaxed configuration is given by:

$$D_{III}^P = \dot{\gamma}^P M, \quad (65)$$

where,

$$M = \frac{1}{\sqrt{2}\tau} R_{III}^{eT} [T - \frac{1}{J} F_{III}^e \bar{B} F_{III}^{eT}] R_{III}^e; \quad (66)$$

where \bar{B} is coaxial with the left plastic stretch of F_{III}^P , $V_{III}^P = R_{III}^P U^P R_{III}^{PT}$; $T = \frac{1}{J} \mathcal{L}^e[\ln V^e]$; and, therefore, M is the direction of the driving stress state in the relaxed configuration, F_{III}^P . We note that the principal values of \bar{B} are identical to those of \tilde{B} given above in equation (61).

A schematic of the deforming glassy polymer is shown in Figure 5. Here, the relaxed configuration of both representations II and III are shown. We note that the two relaxed configurations simply differ by a rotation, where we have:

$$F_{II}^P = R U^P; \quad (67)$$

$$F_{III}^P = R_{III}^P U^P; \quad (68)$$

which results in:

$$\mathbf{F}_{II}^p = \mathbf{R}_{III}^e \mathbf{F}_{III}^p. \quad (69)$$

The corresponding spin of these two relaxed configurations are given by:

$$\mathbf{L}_{III}^p \equiv \mathbf{D}_{III}^p = \dot{\mathbf{F}}_{III}^p \mathbf{F}_{III}^{p-1}; \quad (70)$$

and

$$\mathbf{L}_{II}^p \equiv \mathbf{D}_{II}^p + \mathbf{W}_{II}^p = \dot{\mathbf{R}}_{III}^e \mathbf{R}_{III}^{eT} + \mathbf{R}_{III}^e \mathbf{D}_{III}^p \mathbf{R}_{III}^{eT}. \quad (71)$$

Therefore, $\mathbf{W}_{II}^p = \dot{\mathbf{R}}_{III}^e \mathbf{R}_{III}^{eT}$. This connection between the two relaxed configurations for the glassy polymer is analogous to that presented in section 2 for the single crystal. A detailed proof of the independence of the solution upon the kinematic decomposition due to the isotropic character of both the Cauchy stress and the back stress is given in the Appendix.

Identical results are, of course, obtained using the kinematic decomposition of representations II and III for the glassy polymer. In particular, the analysis of the simple shear of a glassy polymer, polymethylmethacrylate, was conducted. The functions for $\dot{\gamma}^p$ and \mathbf{B} and all necessary material properties are given in BOYCE, ET AL.. The resulting shear stress-displacement curves for both *R-II* and *R-III* are shown in Figure 6. The solutions are identical.

4 Conclusion

This paper has examined the representation of the kinematics of inelastic deformation problems involving finite strains and rotations. The independence of problem solution on kinematic representation was demonstrated using two distinct examples. The first example considered the planar single crystal of PIERCE, ET AL. subjected to tensile loading. The second example analyzed the glassy polymer as constitutively modelled by BOYCE, ET AL. subjected to simple shear loading. Both problems were solved utilizing various representations of the kinematics. The solutions were found to be independent of the choice of the relaxed configuration, i.e. the representation, as shown in Figures 3 and 6. The two cases discussed in this paper were examined with physically-based constitutive models, where the tensorial representation of the resistances to deformation in the single crystal and the glassy polymer are, in a sense, clearly understood. The tensorial state representation of the physics of deformation is not, at this point, as well characterized for more complex materials such as polycrystalline metals. This has

lead to the corresponding debate over the representation of the kinematics of plastic deformation in the presence of anisotropy and the role of the "plastic spin".

It can be concluded that the most convenient representation for the material of concern may be chosen when solving problems involving finite strain plasticity. However, on reflection, one also wishes to acknowledge that it is of great interest to retain information on the state and orientation of the material. General representations which monitor this information with internal state variables are highly desirable. General representations also permit a greater freedom for improvement or expansion of existing material models by providing a kinematic framework independent of material model which will not require modification to accomodate material law modifications.

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REFERENCES

- | | | |
|-----------------------------|------|--|
| AGAH-TEHRANI, A., LEE, E.H. | 1987 | <i>J. Mech. Phys. Solids</i> , 35 , p.519. |
| MALLETT, R.L. and E.T. ONAT | | |
| ANAND, L. | 1979 | <i>J. Appl. Mech.</i> , 46 , 78. |
| ANAND, L. | 1985 | <i>Int. J. of Plasticity</i> , 1 , 213. |
| ASARO, R.J. | 1983 | in <i>Advances in Applied Mechanics</i> , 23 , Academic Press, NY. |
| BOYCE, M.C., PARKS, D.M. | 1988 | <i>Mech. of Matls.</i> , in press. |
| and A.S. ARGON | | |
| DAFALIAS, Y.F. | 1984 | <i>Mech. of Matls.</i> , 3 , 223. |
| DAFALIAS, Y.F. | 1985 | <i>J. Appl. Mech.</i> , 52 , 825. |
| FARDSHISHEH, F. | 1972 | in <i>Problems in Plasticity</i> , ed. A. Sawczuk, Noordhoff, Leyden. |
| and E.T. ONAT | | |
| HAVNER, K.S. | 1982 | in <i>Mechanics of Solids: The Rodney Hill 60th Ann. Vol.</i> , eds. Hopkins, H.G., Sewell, M.J., Pergamon Press |
| HILL, R. and J.R. RICE | 1972 | <i>J. Mech. Phys. Solids</i> , 20 , 401. |
| LEE, E.H. | 1969 | <i>J. Appl. Mech.</i> , 36 , 1. |
| LORET, B. | 1983 | <i>Mech. of Matls.</i> , 2 , 287. |
| ONAT, E.T. | 1982 | Ch. 5 of <i>Recent Advances in Creep and Fracture of Engineering Materials and Structures</i> , Pineridge Press, Swansea, U.K. |
| ONAT, E.T. | 1987 | "Representation of Elastic-Plastic Behavior in the Presence of Finite Deformations and Anisotropy", submitted to <i>Int. J. Plasticity</i> . |
| PARKS, D.M., ARGON, A.S. | 1984 | MIT Program in Polymer Science and Technology Report. |
| and B. BAGEPALLI | | |
| PIERCE, D., ASARO, R.J. | 1983 | <i>Acta Metall.</i> , 31 , 1951. |
| and A. NEEDLEMAN | | |

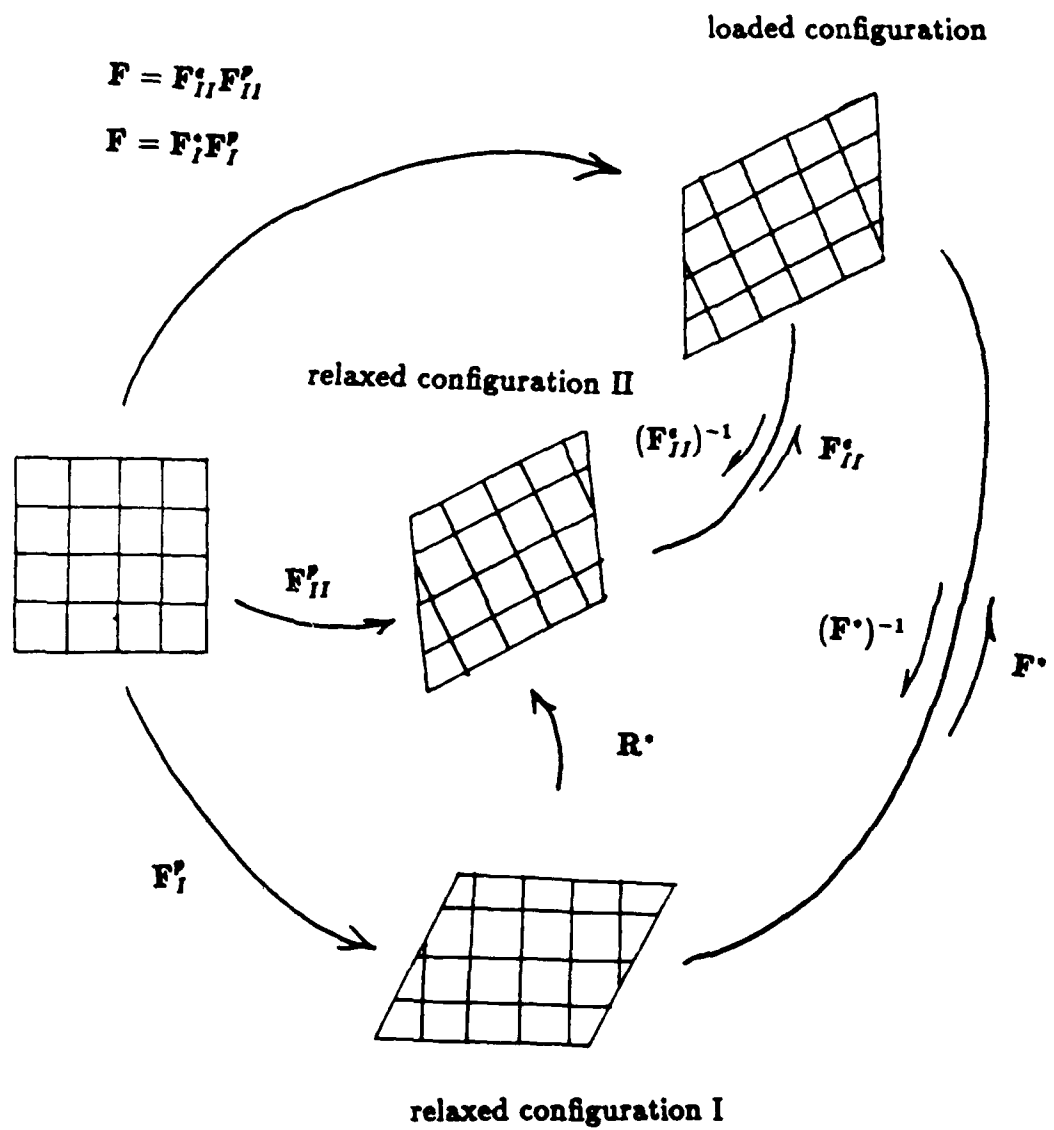


Figure 1. Schematic of a deforming body depicting the "unloaded" states as described by representations I and II.

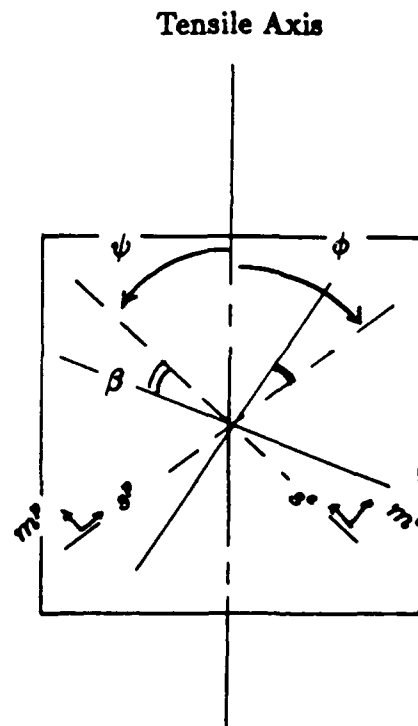


Figure 2. Schematic of the planar single crystal containing two slip systems. The primary and conjugate systems are depicted with respect to the tensile load axis.

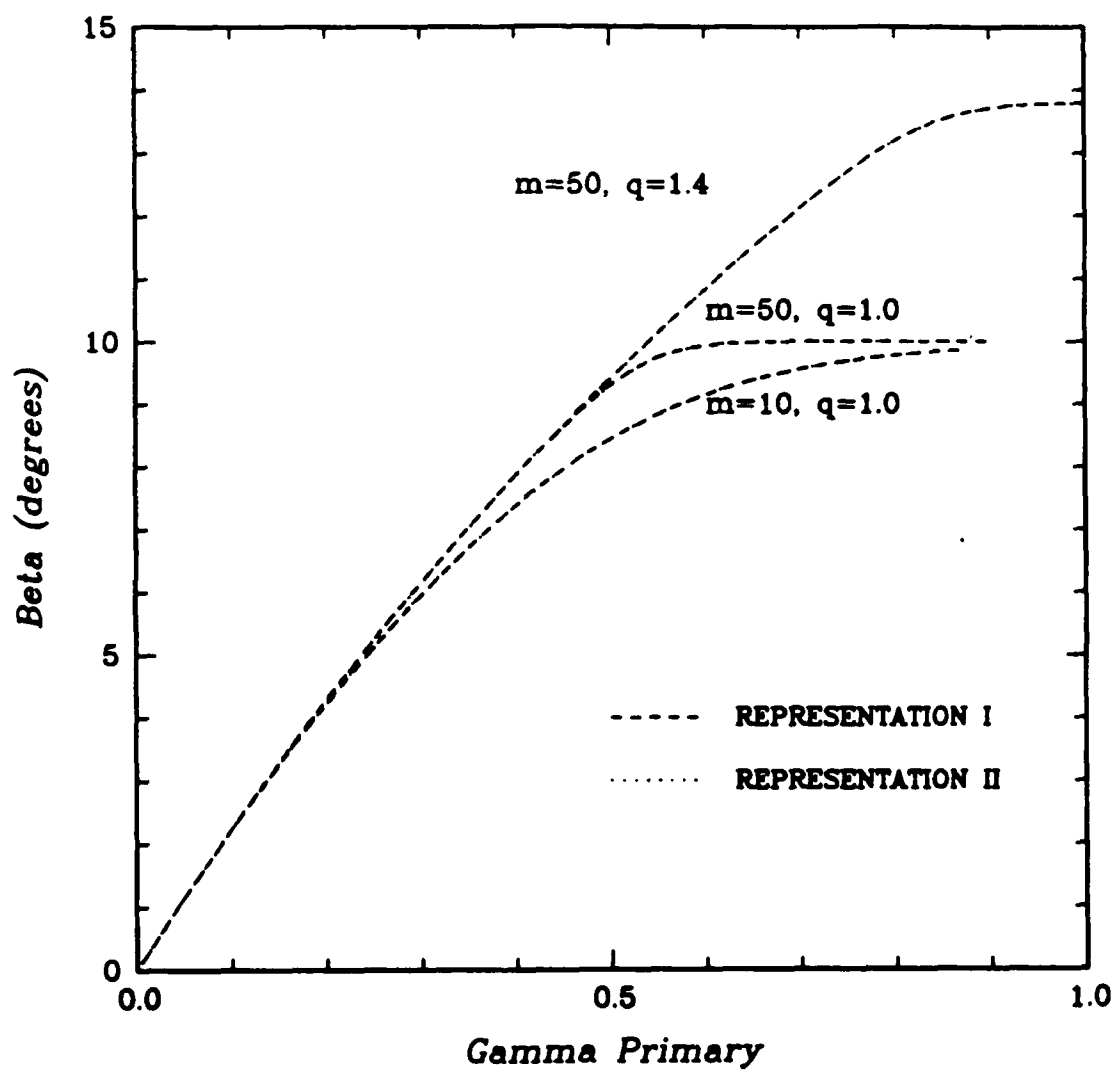


Figure 3. Lattice orientation, β , vs primary plastic shear, γ^p , using both representations I and II. Results are identical.

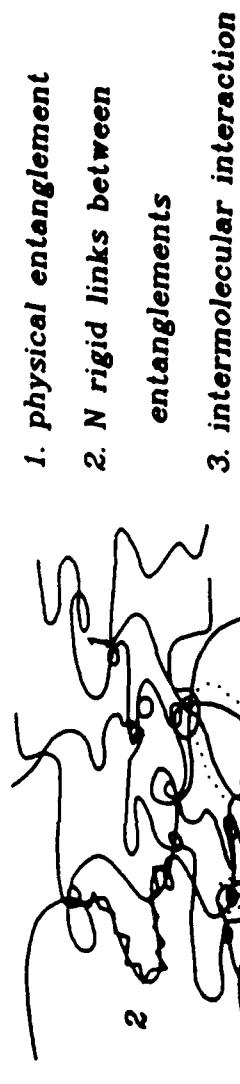


Figure 4. Schematic of amorphous polymer.

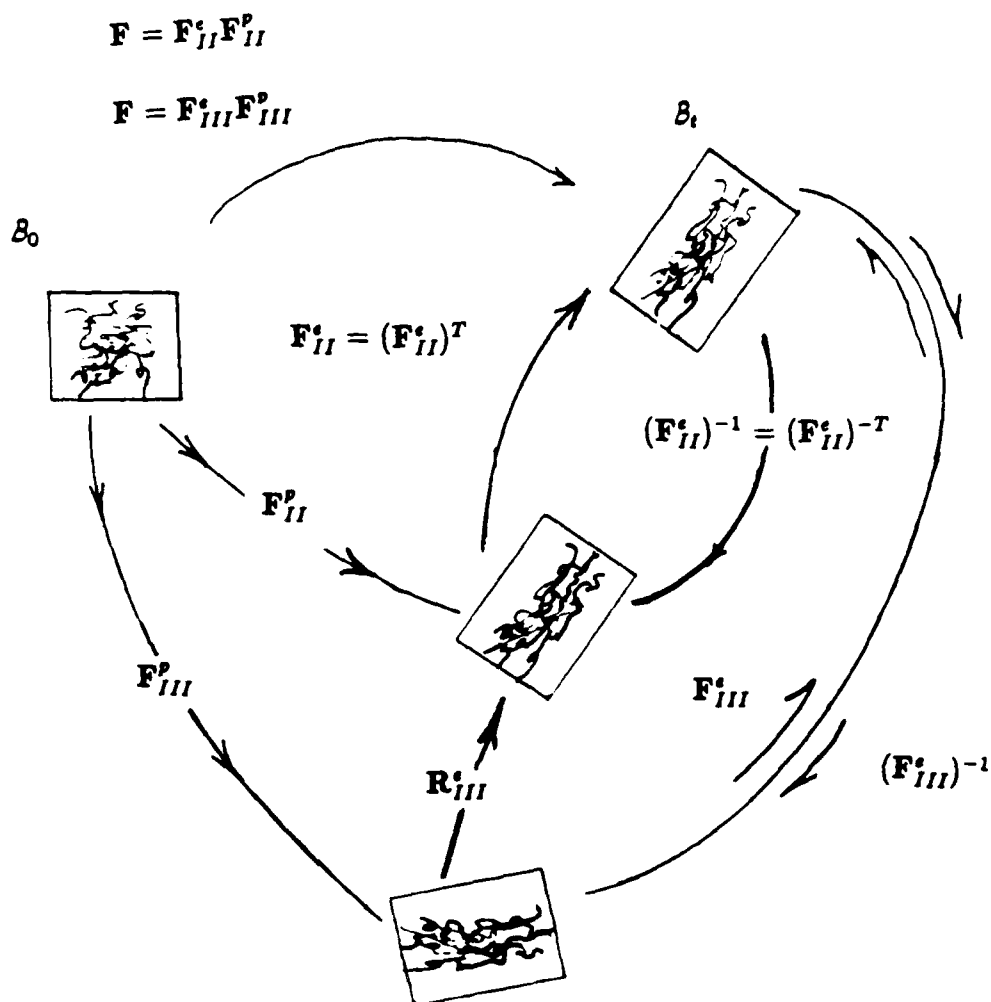


Figure 5. Schematic of a deforming glassy polymer depicting relaxed configurations II and III.

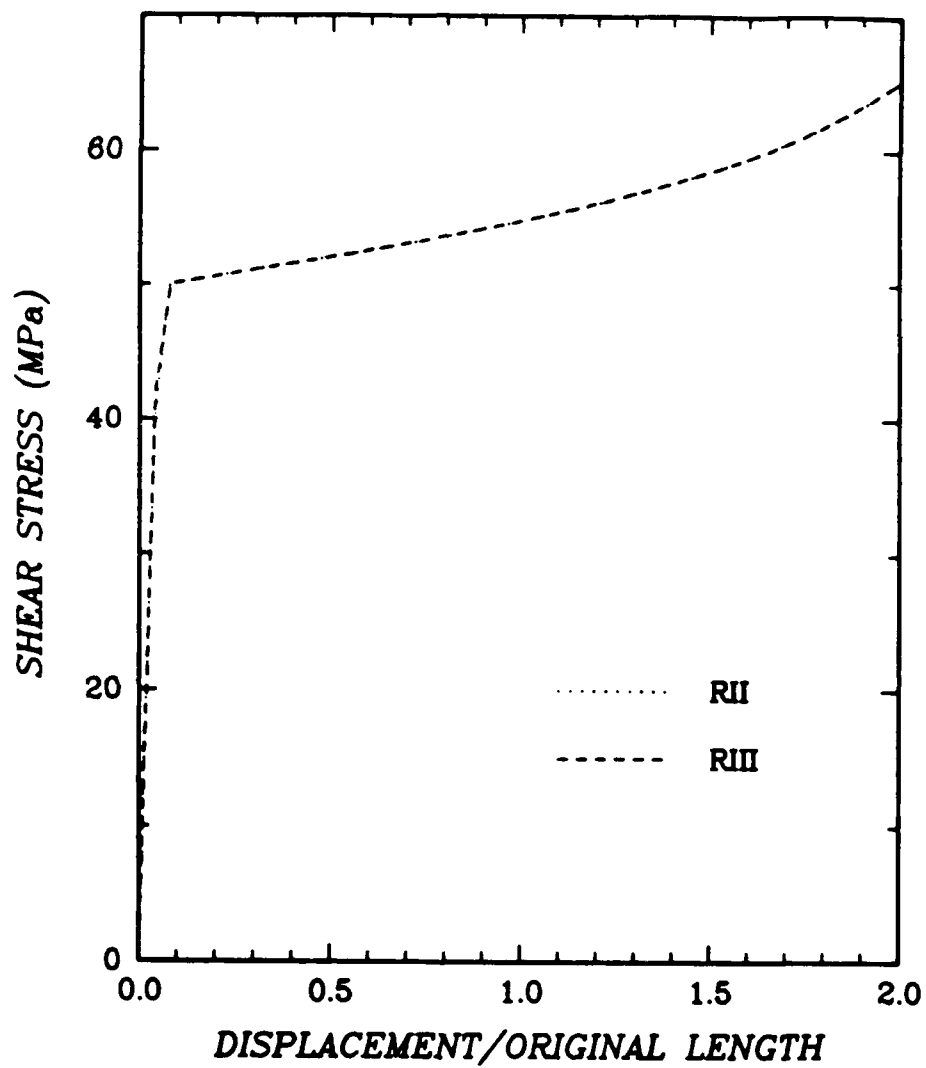


Figure 6. Simple Shear of PMMA -- Shear Stress vs. Normalized Displacement using kinematic representations II and III. Results are identical.

Appendix A

In section 3 of this paper, a symmetric form for the plastic velocity gradient was postulated. The resulting decomposition was referred to as representation III, *R-III*. It is the purpose of this appendix to show that, in general, the introduction of an arbitrary skew-symmetric component of the plastic velocity gradient tensor does not affect the Cauchy stress material response as long as the remaining constitutive equations are form invariant as in the case of the glassy polymer. In the paper, such an example was analyzed, where the "W^p" which was an algebraic result of the symmetry imposed upon \mathbf{F}^p was the "arbitrary" skew-symmetric component. The corresponding decomposition was referred to as representation II, *R-II*.

For the proof, we consider two constitutive initial value problems, where the only difference is the flow rule:

Case a) The plastic velocity gradient as expressed in the relaxed configuration is taken to be symmetric (*RIII* of the paper).

$$\begin{aligned} \mathbf{L}_{III}^p &\equiv \mathbf{D}^p(\bar{\mathbf{T}}, \bar{\mathbf{B}}, \bar{s}); \\ \dot{\bar{\mathbf{F}}}_{III}^p &= \mathbf{D}^p(\bar{\mathbf{T}}, \bar{\mathbf{B}}, \bar{s}) \bar{\mathbf{F}}_{III}^p; \\ \dot{\bar{s}} &= g(\bar{\mathbf{T}}, \bar{\mathbf{B}}, \bar{s}); \\ \bar{\mathbf{T}} &= \mathcal{L}^s[\ln \mathbf{U}_{III}^s]; \\ \bar{\mathbf{B}} &= \mathbf{f}(\mathbf{V}_{III}^p); \end{aligned} \tag{A.1}$$

with the initial conditions of $\bar{\mathbf{F}}_{III}^p(0) = \mathbf{I}$, (and, therefore, $\bar{\mathbf{B}}(0) = 0$) and $\bar{s}(0) = s_0$. The overlaid bar indicates that these are expressions for the description of state in the relaxed configuration, $\bar{\mathbf{F}}_{III}^p$, i.e. not in the loaded configuration. Note that \mathbf{D}^p , g , \mathcal{L}^s , and \mathbf{f} are isotropic functions.

Case b) The plastic velocity gradient is taken to be arbitrary. (One such example was *R-II* given in the body of this paper.)

$$\begin{aligned}
\mathbf{L}_i^p &= \mathbf{D}^p(\tilde{\mathbf{T}}, \tilde{\mathbf{B}}, \tilde{s}) + \mathbf{W}_i^p; \\
\dot{\mathbf{F}}_i^p &= [\mathbf{D}^p(\tilde{\mathbf{T}}, \tilde{\mathbf{B}}, \tilde{s}) + \mathbf{W}_i^p] \mathbf{F}_i^p; \\
\dot{\tilde{s}} &= \mathbf{g}(\tilde{\mathbf{T}}, \tilde{\mathbf{B}}, \tilde{s}); \\
\tilde{\mathbf{T}} &= \mathcal{L}^e[\ln \mathbf{U}_i^e]; \\
\tilde{\mathbf{B}} &= \mathbf{f}(\mathbf{V}_i^p);
\end{aligned} \tag{A.2}$$

with the initial conditions of $\mathbf{F}_i^p(0) = \mathbf{I}$, (and, therefore, $\tilde{\mathbf{B}}(0) = 0$) and $\tilde{s} = s_0$. The overlaid tilde indicates that these are expressions for the description of state in the relaxed configuration, \mathbf{F}_i^p , i.e. not in the loaded configuration.

In order to prove our assertion that the system of equations in cases (a) and (b) will lead to identical solutions, we consider a "trial" solution for the second system in the form:

$$\begin{aligned}
\mathbf{F}_i^p &= \mathbf{Q} \mathbf{F}_{III}^p; \\
\tilde{s} &= \bar{s};
\end{aligned} \tag{A.3}$$

where \mathbf{Q} is a rotation tensor to be determined. From these relations, we obtain:

$$\begin{aligned}
\mathbf{R}_i^e &= \mathbf{R}_{III}^e \mathbf{Q}^T; \\
\mathbf{U}_i^e &= \mathbf{Q} \mathbf{U}_{III}^e \mathbf{Q}^T; \\
\mathbf{V}_i^p &= \mathbf{Q} \mathbf{V}_{III}^p \mathbf{Q}^T.
\end{aligned} \tag{A.4}$$

Due to the isotropic character of \mathbf{f} and \mathcal{L}^e , the above relations yield:

$$\begin{aligned}
\ln(\mathbf{U}_i^e) &= \mathbf{Q} \ln(\mathbf{U}_{III}^e) \mathbf{Q}^T; \\
\tilde{\mathbf{B}} &= \mathbf{Q} \bar{\mathbf{B}} \mathbf{Q}^T; \\
\tilde{\mathbf{T}} &= \mathbf{Q} \bar{\mathbf{T}} \mathbf{Q}^T.
\end{aligned} \tag{A.5}$$

Next, differentiating \mathbf{F}_i^p as given in equation (A3), and then substituting in the flow rules of cases (a) and (b), we obtain:

$$\dot{\mathbf{F}}_i^p \mathbf{F}_i^{p-1} = \dot{\mathbf{Q}} \mathbf{Q}^T + \mathbf{Q} \mathbf{D}^p(\tilde{\mathbf{T}}, \tilde{\mathbf{B}}, \tilde{s}) \mathbf{Q}^T = \mathbf{D}^p(\tilde{\mathbf{T}}, \tilde{\mathbf{B}}, \tilde{s}) + \mathbf{W}_i^p. \tag{A.6}$$

Due to (A5) and the symmetry and isotropic character of \mathbf{D}^p , equation (A6) is satisfied provided:

$$\dot{\mathbf{Q}} \mathbf{Q}^T = \mathbf{W}_i^p. \tag{A.7}$$

From equations (A4) and (A5), we also verify that:

$$(\det \mathbf{U}_i^e)^{-1} \mathbf{R}_i^e \tilde{\mathbf{T}} \mathbf{R}_i^{eT} = (\det \mathbf{U}_{III}^e)^{-1} \mathbf{R}_{III}^e \bar{\mathbf{T}} \mathbf{R}_{III}^{eT}. \tag{A.8}$$

Therefore, the Cauchy stress tensor \mathbf{T} obtained by integration of one or the other system of constitutive equations is identical.